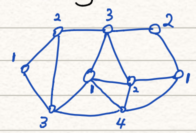
Colouring



4 - colouring

Defn: A K-colouring of G is an assignment of a colour to each vertex using a pool of at most K colours
A graph has a k-colouring is k- colourable.

Note: if G is k-colourable, then G is (k+1)-colourable.

General colouring question: How many colours are needed to colour a graph? we want to minimize this.
Applications: ① scheduling
② compilers assigning variables to registers

Suppose G has n vertices ⇒ G is n colorable

$\boxed{\text{Thm: } K_n \text{ is n-colourable, but not (n-1)-colourable}}$
complete graph

$\boxed{\text{Thm: G is 2 colourable iff G is bipartite}}$

Colouring planar graph

Theorem: Any planar has a vertex of degree at most 5.

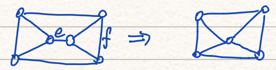
proof: suppose G is planar with n vertices.
Suppose every vertex in G has degree at least 6. Then G has at least $\frac{6n}{2} = 3n$ edges (By handshaking)
Since it is a planar graph, it has at most 3n-6 edges. ⇒ contradictions.
So at least one vertex has degree at most 5.
■

Theorem: Any planar graph is 6-colourable.

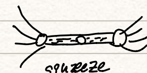
proof: by induction on the # of vertices n.
B.C. n=1. then it is 6-colourable
I.H: Any planar graph with n-1 vertices is 6-colourable,
I.S: suppose G is planar with n vertices.
Let v be a vertex of degree at most 5 in G. Obtain G' from G by removing v and its incident edges.
Then G' is planar with n-1 vertices.
By induction hypothesis, we know G' is 6-colouring.
Keep the same colouring for G. For v, at most 5 colours are used by its neighbour.
Since we have 6 colours, at least one unused colour is available for v.
We get a 6-colouring for G.
■

Theorem: Any planar graph is 5-colourable

contraction G/e



if G is planar, then G/e is also planar.



squeeze together

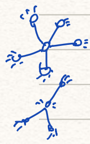
proof of 5-colour thm. strong induction on the number of vertices n.

Base case: Any planar graph with at most 5 vertices is 5-colourable

after contraction, planarity needs to hold.

I.H. Assume any planar graph with fewer than n vertices is 5-colourable.

I.S. Suppose G is a planar graph with n vertices.



Let v be a vertex of degree at most 5.

if $\deg(v) \leq 4$, then apply the argument from 6-color theorem to prove that it is 5-colourable.

Assume that $\deg(v) = 5$. There exist two neighbours x, y of v that are not adjacent. for otherwise G contains K_5 which is not planar.

Let G' be obtained from G by contracting vx, vy. Let v^* be the contracted vertex.

Now G' is planar with $n-2$ vertices. By induction G' is 5-colourable.

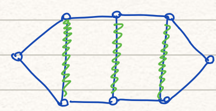
We keep the same colours for G except x, y receive the colour of v^* (this is possible since $x y$ are not adjacent) So the neighbours of v use at most 4 colours.

Since there are 5 colours at least 1 is available for v.

$\square$

Theorem. Every planar graph is 4-colourable. ?!?!?! (compute prove ^^)

Matching

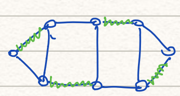


A matching in G is set of edges where no two edges share a common $\boxed{\text{vertex}}$ (one edge also a matching)

(each vertex is incident with at most one edge in a matching)

~ matching (empty set is a matching)

General Q: what is the maximum size of a matching in a graph?



~ perfect matching. (matching every single vertex)

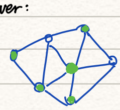
perfect match ⇒ maximum matching

Defn: A vertex incident within an edge in a matching is saturated. Otherwise it is unsaturated.

A matching that saturates every vertex is a perfect matching.

(odd graph does not have perfect matching)

Cover:



• → cover

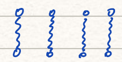
Defn: A cover C of a graph G is a set of vertices where every edge in G has at least one endpoint in C.

General Q: what is the minimum size of a cover in a graph?

Matching vs covers

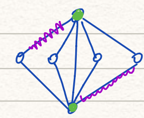
Suppose M is a matching and C is a cover

e.g.



$\Rightarrow$ minimum # cover = 4 (sharing no common vertex (matching))

$|C| \geq |M|$



matching size 2

cover size 2

Thm: If M is a matching and C is a cover of G, then $|M| \leq |C|$. (if $|M| = |C|$, then we find the max matching & min cover)

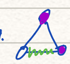
proof:

For each edge uv in M, at least one of u or v in C. Since edges in M do not share common vertices, C must contain at least $|M|$ distinct vertices. $\square$

Corollary: if M is matching and C is a cover of G where $|M| = |C|$, then M is a maximum matching and C is a minimum cover.

proof: Let M' be any matching. Then $|M'| \leq |C| = |M|$, so M is maximum.
Let C' be any cover. Then $|C'| \geq |M| = |C|$, so C is minimum. $\square$

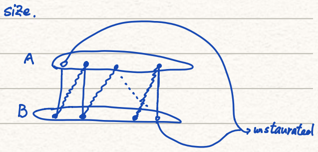
In general, size of max matching might not be equal to size of min. cover. e.g.

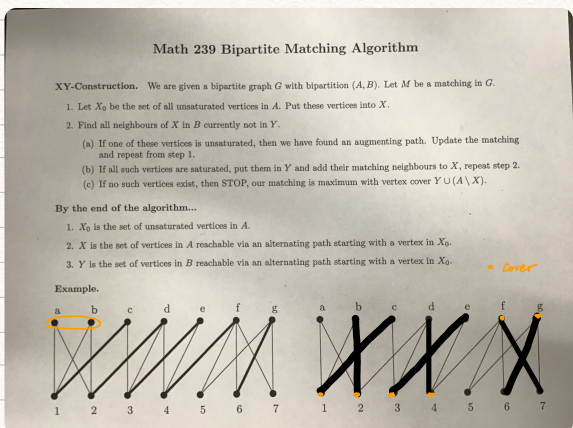


König's Theorem:

In a bipartite graph, the size of a maximum matching is equal to the size of a minimum cover.

idea: attempt to find an argumenting path
if find one, update matching. if fail to find one, the matching is maximum and find a cover of same size.





Math 239 Bipartite Matching Algorithm

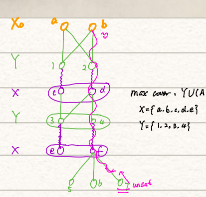
XY-Construction. We are given a bipartite graph G with bipartition (A, B). Let M be a matching in G.

1. Let X_0 be the set of all unsaturated vertices in A. Put these vertices into X.
2. Find all neighbours of X in B currently not in Y.
   (a) If one of these vertices is unsaturated, then we have found an augmenting path. Update the matching and repeat from step 1.
   (b) If all such vertices are saturated, put them in Y and add their matching neighbours to X, repeat step 2.
   (c) If no such vertices exist, then STOP, our matching is maximum with vertex cover $Y \cup (A \setminus X)$.

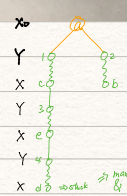
By the end of the algorithm...

1. X_0 is the set of unsaturated vertices in A.
2. X is the set of vertices in A reachable via an alternating path starting with a vertex in X_0.
3. Y is the set of vertices in B reachable via an alternating path starting with a vertex in X_0.

Example.

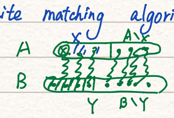


max cover: $Y \cup (A \setminus X)$
$X = \{a, b, c, d, e\}$ $\Rightarrow$ cover: $\{1, 2, 3, 4, f, g\}$
$Y = \{1, 2, 3, 4\}$



$\Rightarrow$ stuck $\Rightarrow$ maximum match & cover, can not find anything longer.

Proof: Suppose M, X_0, X, Y are sets we get at the end of the bipartite matching algorithm



and $B \setminus Y$.

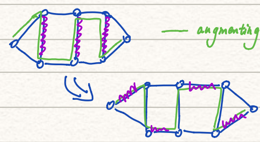
We see that there is no edge between X and $B \setminus Y$. Since the algorithm would discover this edge and place the vertex in $B \setminus Y$ in Y.
Thus we claims that $Y \cup (A \setminus X)$ is a cover
Now every vertex in Y is saturated, for otherwise the algorithm finds an augmenting path, and would have continued.
also every vertex in $A \setminus X$ is saturated, since all unsaturated vertices in A are in X_0
No matching edges joins Y to $A \setminus X$ (since matching neighbor of Y are in X)
So each vertex in $Y \cup (A \setminus X)$ is saturated by a distinct matching edges.
Then $|M| = |Y \cup (A \setminus X)|$

$\blacksquare$

Corollary: A bipartite graph G with m edges and maximum degree d has a matching of size at least m/d.

proof: we just need to prove that every cover has size at least m/d (by Konig's Thm)
Let C be a cover. Each vertex in C covers at most d edges. Then C covers at most $d|C|$ edges.
But $d|C| \geq m$. So $|C| \geq \frac{m}{d}$ $\square$

Augmenting paths:



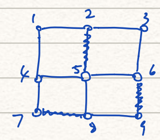
— augmenting path (both end with unsaturated vertices) ⇒ matching edge & unmatching edge appear alternatively.

(switch edges in M) if not finding augmenting path ⇒ we claim that it is the maximum matching

Defn: An alternating path P with respect to a matching M is a path where consecutive edges alternate between in M and not in M

An alternating path is an alternative path that starts and ends with different unsaturated vertices (the first & last one are not in M)

e.g.



1, 4 is an alternating path

☆ If an augmenting path exist, then there is a larger matching.

⇒ aug. path 

⇒ 1 + even + 1 ⇒ even # of vertices ⇒ odd path. ⇒ switch get one larger.

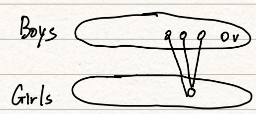
↳ if $e \in M$, remove e from; if $e \notin M$, add e to M

Hall's theorem:

When does a max matching exist that saturates all of A?

What prevents us from saturating all of A?

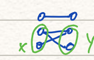
|Boys| > |girls|



deg(v) = 0

$X \subseteq$ Boys, $|N(X)| < |X|$ ↳ neighbour of X

Defn: Let X be a set of vertices in G. The neighbour set N(X) is the set of all vertices in G adjacent to at least one vertex in X.



Hall's theorem. Let G be bipartite with bipartition (A, B). Then G has a matching that saturates every vertex in A iff for all $D \subseteq A$, $|N(D)| \geq |D|$

proof: ⇒ suppose M is the matching that saturates A. Let $D \subseteq A$. Then each vertex in D is matched to a distinct vertex in N(D) using M. So $|N(D)| \geq |D|$

⇐ suppose a maximum matching M does not saturate every vertex in A. Then $|M| < |A|$

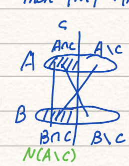
(Goal: Find a $D \subseteq A$ where $|N(D)| < |D|$).

By König's Theorem. there exists a cover C where $|M| = |C|$

Since C is a cover, there is no edge between $A \setminus C$ and $B \setminus C$.

So $N(A \setminus C) \subseteq B \setminus C$. $|C| = |A \cap C| + |B \cap C| = |M| < |A|$

Then $|N(A \setminus C)| \leq |B \cap C| < |A| - |A \cap C| = |A \setminus C|$. So $|N(A \setminus C)| < |A \setminus C|$ □



Corollary: A k-regular bipartite graph with $k \geq 1$ has a perfect matching.

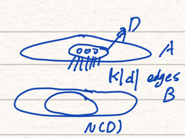
proof: Suppose G has bipartition (A, B). Let $D \subseteq A$

There are $k|D|$ edges incident with D, and the other endpoints are in N(D).

There are $k \cdot |D|$ edges incident with N(D)

So $k|D| \leq k|N(D)|$, since $k \geq 1$. thus $|D| \leq |N(D)|$

By Hall's theorem, there exist a matching that saturates A. So B is also saturated. Hence it is a perfect matching.



Corollary: Any k-regular bipartite graph can be partitioned into k perfect matching.